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Computing option pricing models under transaction costs ☆

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ABSTRACT

This paper deals with the Barles–Soner model arising in the hedging of portfolios for option pricing with transaction costs. This model is based on a correction volatility function Ψ solution of a nonlinear ordinary differential equation. In this paper we obtain relevant properties of the function Ψ which are crucial in the numerical analysis and computing of the underlying nonlinear Black–Scholes equation. Consistency and stability of the proposed numerical method are detailed and illustrative examples are given.

1. Introduction

In a complete financial market without transaction costs, the Black–Scholes (B–S) no-arbitrage argument provides a rational option pricing formula and a hedging portfolio that replicates the contingent claim. Under the transaction costs, the continuous trading required by the hedging portfolio is prohibitively expensive, [1]. Several alternatives lead to option prices that are equal to Black–Scholes price but with an adjusted volatility. In 1992, Boyle and Vorst [2], derived from a binomial model an option price taking into account transaction costs and that is equal to a B–S price but with a modified volatility of the form

$$\sigma = \sigma_0(1+cA)^{1/2}, \quad A = \frac{\mu}{\sigma_0\sqrt{\Delta T}}, \; c = 1.$$

Here, μ is the proportional transaction cost, Δt the transaction period, and σ_0 is the original volatility constant. Leland [3] computed $c = (\frac{2}{\pi})^{1/2}$. Kusuoka [4] then showed that the "optimal" c depends on the risk structure of the market. Paras and Avellaneda [5] derived the modified volatility

$$\sigma = \sigma_0(1 + A \,\mathrm{sign}\,(V_{SS}))^{1/2},$$

from a binomial model using the algorithm of Bensaid et al. [6]. Whalley and Wilmott [7] using an asymptotic analysis based on [8] propose the same adjusted volatility. A comparison of the exact hedging strategy of [8] and the asymptotic hedging strategy of [7] has been studied in [9]. Here, V is the option price, S the price of the underlying asset, and V_{SS} denotes the second derivative of V with respect to S (the "Gamma"). In particular, the option price does not need to be convex.

Kratka in [10] and Jandačka and Ševčovič in [11] propose a correction of volatility of the form

$$\sigma^2 = \sigma_0^2(1+\mu(S\,V_{SS})^{\frac{1}{3}}),$$

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where $\mu = 3(C^2R/2\pi)^{\frac{1}{3}}$ and C, R are nonnegative constants representing the transaction cost measure and the risk premium measure, respectively.

A more complex model has been proposed by Barles and Soner [1], assuming that investor's preferences are characterized by an exponential utility function. In their model the nonlinear volatility reads

$$\sigma^2 = \sigma_0^2(1 + \Psi[\exp(r(T-t)a^2S^2V_{SS})]), \tag{1.1}$$

where r is the risk-free interest rate, T the maturity, and $a = \mu\sqrt{\gamma N}$, with risk aversion factor γ and the number N of options to be sold. The function Ψ is the solution of the nonlinear initial value problem

$$\Psi'(A) = \frac{\Psi(A)+1}{2\sqrt{A\Psi(A)} - A}, \quad A \neq 0,\ \Psi(0) = 0. \tag{1.2}$$

In the mathematical literature, only a few results can be found on the numerical discretization of B–S equation, mainly for linear B–S equations. The numerical approaches vary from finite element discretizations [12,13], finite-difference approximations [14–16]. The numerical discretization of the B–S equations with the nonlinear volatility (1.2) has been performed using explicit finite-difference schemes [1]. However, explicit schemes have the disadvantage that restrictive conditions on the discretization parameters (for instance, the ratio of the time and the space step) are needed in order to obtain stable, convergent schemes [17]. Moreover, the order of convergence is only one in time and two in space. [18] combines high-order compact difference schemes derived by [19] and techniques to construct numerical solutions with frozen values of the nonlinear coefficient of the nonlinear B–S equation to make the formulation linear.

In this paper we use a semidiscretization technique by using fourth-order difference approximations of the partial derivatives V_S and V_{SS} arising in the nonlinear B–S equation

$$V_t + \frac{1}{2}\sigma(V_{SS})^2S^2V_{SS} + rSV_S - rV = 0. \tag{1.3}$$

Then we achieve an ordinary system of nonlinear ordinary differential equations with respect to the time, that is solved numerically. Apart form (1.3), in the Barles–Soner model one has the terminal condition

$$V(S,T) = \max(0, S-E), \quad S > 0, \tag{1.4}$$

and the boundary conditions

$$V(0,t) = 0, \quad \lim_{s\to\infty} \frac{V(S,t)}{S - Ee^{-r(T-t)}} = 1. \tag{1.5}$$

In order to compute the numerical solution, it is necessary to work in a bounded domain. Once this numerical domain has been chosen, the boundary conditions can be translated from the asymptotic condition (1.5), as it is done for instance in [1] or [18], or the boundary values must be found together with the solution and they are linked with the rest of the numerical solution in the interior of the numerical domain by using extrapolation techniques. This last approach is used in this paper in accordance with the used scheme.

Using the change of variable $\tau = T - t$, $U(S,\tau) = V(S,t)$ Eq. (1.3) together with the initial condition (1.4) is transformed into

$$U_\tau - \frac{S^2}{2}\sigma^2U_{SS} - rSU_S + rU = 0, \quad 0 < S < \infty,\ 0 < \tau \le T, \tag{1.6}$$

$$U(S,0) = \max(0, S-E). \tag{1.7}$$

This paper is organized as follows. Section 2 is addressed to the study of the properties of the volatility correction function Ψ after obtaining the implicit solution of (1.2). In Section 3, by using semidiscretization with respect to S one gets a nonlinear system of ordinary differential equations with respect to the time, and then it is discretized using a forward explicit scheme. This approach allows us to study the stability and consistency of the nonlinear scheme in Sections 4 and 5 without using linearization strategies as it is done in [18]. Section 6 includes illustrative examples of European call option pricing where the computed numerical solution and their properties are checked.

If A is a matrix in $\mathbb{R}^{p\times p}$ and A^t denotes its transposed matrix, we denote by $\|A\|$ the spectral norm of A defined as, [20],

$$\|A\| = \max\left\{\sqrt{\lambda};\ \lambda \text{ eigenvalue of } A^tA\right\}.$$

If q is an integer with $|q| \le p-1$, and A_q is a band matrix in $\mathbb{R}^{p\times p}$ such that $A_q = (a_{ij})$ with $a_{ij} = 0$ everywhere outside of the diagonal $j = i+q$, then it is easy to show that

$$\|A_q\| = \max\left\{|a_{i,i+q}|;\ \begin{matrix} 1 \le i \le p-q & \text{if } q \ge 0 \\ 1-q \le i \le p & \text{if } q < 0 \end{matrix}\right\}. \tag{1.8}$$

2. Properties of the correction of volatility function

We begin this section by showing an exact implicit expression of the solution Ψ of Eq. (1.2) as well as the growth properties and convexity that will play an important role in the numerical analysis of Eq. (1.6).

From Theorem 2.1 of [21] it is known that $\Psi(A)$ is an increasing function mapping the real line onto the interval $]-1, +\infty[$ and $\Psi(A)$ is implicitly defined by

$$A = \left(-\frac{\operatorname{Arcsinh}\sqrt{\Psi}}{\sqrt{\Psi+1}} + \sqrt{\Psi}\right)^2, \quad \text{if } \Psi > 0, \tag{2.1}$$

$$A = -\left(\frac{\arcsin\sqrt{-\Psi}}{\sqrt{\Psi+1}} - \sqrt{-\Psi}\right)^2, \quad \text{if } -1 < \Psi < 0. \tag{2.2}$$

Taking derivatives in (2.1) it follows that

$$\frac{\mathrm{d}A}{\mathrm{d}\Psi} = -\left(\frac{\operatorname{Arcsinh}\sqrt{\Psi}}{\Psi+1}\right)^2 + \frac{\Psi}{\Psi+1}, \quad \Psi > 0, \tag{2.3}$$

$$\frac{\mathrm{d}^2A}{\mathrm{d}\Psi^2} = \frac{2(\operatorname{Arcsinh}\sqrt{\Psi})^2}{(\Psi+1)^3} + \frac{1}{\sqrt{\Psi(\Psi+1)^5}}\left(\sqrt{\Psi(\Psi+1)} - \operatorname{Arcsinh}\sqrt{\Psi}\right), \tag{2.4}$$

and evaluating the sign of $\left(\sqrt{\Psi(\Psi+1)} - \operatorname{Arcsinh}\sqrt{\Psi}\right)$ one gets

$$\frac{\mathrm{d}^2A}{\mathrm{d}\Psi^2} > 0, \quad \text{if } \Psi > 0. \tag{2.5}$$

Taking two times derivatives of the inverse function one gets

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}A^2} = -\left(\frac{\mathrm{d}^2A}{\mathrm{d}\Psi^2}\right)\left(\frac{\mathrm{d}\Psi}{\mathrm{d}A}\right)^3 < 0, \quad \text{if } A > 0. \tag{2.6}$$

Hence $\Psi(A)$ is a concave function in $]0, +\infty[$.

Taking two times derivatives in (2.2) it follows that

$$\frac{\mathrm{d}A}{\mathrm{d}\Psi} = \left(\frac{\arcsin\sqrt{-\Psi}}{\Psi+1}\right)^2 + \frac{\Psi}{\Psi+1}, \quad -1 < \Psi < 0, \tag{2.7}$$

$$\frac{\mathrm{d}^2A}{\mathrm{d}\Psi^2} = -\frac{2(\arcsin\sqrt{-\Psi})^2}{(\Psi+1)^3} + \frac{1}{\sqrt{-\Psi(\Psi+1)^5}}\left(\sqrt{-\Psi(\Psi+1)} - \arcsin\sqrt{-\Psi}\right), \tag{2.8}$$

and hence,

$$\frac{\mathrm{d}^2A}{\mathrm{d}\Psi^2} < 0, \quad -1 < \Psi < 0. \tag{2.9}$$

Taking two times derivatives of the inverse function and using (2.9) one gets

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}A^2} = -\left(\frac{\mathrm{d}^2A}{\mathrm{d}\Psi^2}\right)\left(\frac{\mathrm{d}\Psi}{\mathrm{d}A}\right)^3 > 0, \quad A < 0. \tag{2.10}$$

Hence, $\Psi(A)$ is a convex function in $]-\infty, 0[$.

Remark 1. From a computational point of view, implicit expressions (2.1) and (2.2) are useful because if one is interested in computing $\Psi(A)$, then using (2.1) for a given value $A > 0$ one solves Eq. (2.1) for Ψ using MATLAB. In order to compute $\Psi(A)$ for a given value $A < 0$, one solves Eq. (2.2) for Ψ using Matlab.

The properties of function Ψ studied in the next result will be used in the consistency and stability analysis of the numerical solution of Eq. (1.6). In particular, although $\Psi(A)$ is not differentiable at $A = 0$, the function $g(A) = A\,\Psi(A)$ is continuously differentiable in the real line.

Lemma 1. *Let $\Psi(A)$ be the nonlinear volatility correction function implicitly defined by (2.1), (2.2) and let $g(A) = A\,\Psi(A)$. Then*

(i)
$$|\Psi(A)| \leq \max\{1, \Psi(|A|)\}, \quad A \in \mathbb{R}. \tag{2.11}$$

(ii) *If* $A > 0$, $A_2 = \left(\sinh\, 2 - \frac{2}{\sqrt{(\sinh\ 2)^2+1}}\right)^2 \simeq 9.58$, *and* $d_2 = \Psi(A_2) - \Psi'(A_2)\, A_2 \ \simeq\ 2.62$ *then*
$$|\Psi(A)| \leq \Psi'(A_2)A + d_2. \tag{2.12}$$

(iii) *If* $A < 0$, $A_1 = -\frac{(4\pi-3\sqrt{3})^2}{36} \simeq -1.51$, *and* $d_1 = \Psi'(A_1)\, A_1 - \Psi(A_1) \ \simeq\ 0.64$ *then*
$$|\Psi(A)| \leq \Psi'(A_1)\ |A| + d_1, \quad A < 0. \tag{2.13}$$

(iv) *Function* $g(A) = A\,\Psi(A)$ *is continuously differentiable at* $A = 0$ *and*
$$|g'(A)| \leq \max\{G,\ 2|A|\Psi'(A_2) + d_2\}, \quad A \in \mathbb{R} \tag{2.14}$$
where
$$G = \max\{|g'(A)|;\, A_1 \leq A \leq A_2\}. \tag{2.15}$$

Proof. (i) This is a direct consequence of Theorem 2.1 of [21].

(ii) By applying Taylor's theorem about A_2 and using (2.6) it follows that
$$\begin{aligned}|\Psi(A)| &= \Psi(A) = \Psi(A_2) + \Psi'(A_2)\ (A - A_2) + \frac{\Psi''(\xi)}{2!}\ (A - A_2)^2 \\ &\leq \Psi(A_2) + \Psi'(A_2)\ (A - A_2),\end{aligned}$$
for some $\xi > 0$ such that $|\xi - A_2| < |A - A_2|$. Thus (ii) is proved.

(iii) By applying Taylor's theorem about A_1 and using (2.10), for some $\xi < 0$ satisfying $|\xi - A_1| < |A - A_1|$, one gets
$$\begin{aligned}|\Psi(A)| &= -\Psi(A) \\ &= -\Psi(A_1) - \Psi'(A_1)\ (A - A_1) - \frac{\Psi''(\xi)}{2!}\ (A - A_1)^2 \leq -\Psi(A) - \Psi'(A_1)\ (A - A_1) \\ &= \Psi'(A_1)\ |A| + \Psi'(A_1)\, A_1 - \Psi(A_1).\end{aligned}$$
This proves (iii).

(iv) Taking into account (2.1), (2.2), (2.3), (2.7) and the following Taylor expansions
$$\left.\begin{aligned}&\mathrm{Arcsinh}\, x + \sqrt{x^2+1}\, x = 2x + O(x^2) \\ &-\mathrm{Arcsinh}\, x + \sqrt{x^2+1}\, x = \frac{2x^3}{3} + O(x^4) \\ &\arcsin\ x + \sqrt{1-x^2}\, x = 2x + O(x^2) \\ &\arcsin\ x - \sqrt{1-x^2}\, x = \frac{2x^3}{3} + O(x^4)\end{aligned}\right\} \tag{2.16}$$
one gets
$$\begin{aligned}\lim_{A\to 0^+} g'(A) &= \lim_{A\to 0^+} A\Psi'(A) = \lim_{\Psi\to 0^+} \frac{A(\Psi)}{A'(\Psi)} \\ &= \lim_{\Psi\to 0^+} (\Psi+1)\frac{-\mathrm{Arcsinh}\,\sqrt{\Psi} + \sqrt{\Psi(\Psi+1)}}{\mathrm{Arcsinh}\,\sqrt{\Psi} + \sqrt{\Psi(\Psi+1)}} = \lim_{\Psi\to 0^+} \frac{\frac{2(\sqrt{\Psi})^3}{3}}{2\sqrt{(\Psi)}} = 0.\end{aligned} \tag{2.17}$$
$$\begin{aligned}\lim_{A\to 0^-} g'(A) &= \lim_{A\to 0^-} A\Psi'(A) = \lim_{\Psi\to 0^-} \frac{A(\Psi)}{A'(\Psi)} \\ &= -\lim_{\Psi\to 0^-} (\Psi+1)\frac{\arcsin\ \sqrt{-\Psi} - \sqrt{-\Psi(\Psi+1)}}{\arcsin\ \sqrt{-\Psi} + \sqrt{-\Psi(\Psi+1)}} = -\lim_{\Psi\to 0^-} \frac{\frac{2(\sqrt{-\Psi})^3}{3}}{2\sqrt{(-\Psi)}} = 0.\end{aligned} \tag{2.18}$$
Hence,
$$\lim_{A\to 0} g'(A) = 0. \tag{2.19}$$
Otherwise
$$g'(0) = \lim_{A\to 0} \frac{g(A) - g(0)}{A} = \lim_{A\to 0} \frac{A\,\Psi(A)}{A} = \Psi(0) = 0. \tag{2.20}$$
From (2.19), (2.20) one gets that $g(A)$ is continuously differentiable at $A = 0$.

Taking into account that $\Psi(A)$ is a concave increasing function for $A > 0$ together with (2.6), (2.12) for $A > A_2$ one gets that $\Psi'(A)$ is decreasing and

$$\begin{aligned} |g'(A)| &= g'(A) = \Psi(A) + A\,\Psi'(A) \le A\,\Psi'(A_2) + d_2 + A\,\Psi'(A_2) \\ &= 2A\,\Psi'(A_2) + d_2, \quad A > A_2. \end{aligned} \tag{2.21}$$

Furthermore, if $A < A_1 < 0$, taking into account that $\Psi(A)$ is a convex increasing function for $A < 0$ and (2.10), (2.13) it follows that $\Psi'(A)$ is increasing for $A < A_1 < 0$ and

$$\begin{aligned} |g'(A)| &\le |\Psi(A)| + |A|\,|\Psi'(A)| \le \Psi'(A_1)|A| + d_1 + |A|\,\Psi'(A_1) \\ &= 2|A|\,\Psi'(A_1) + d_1, \quad A < A_1 < 0. \end{aligned} \tag{2.22}$$

As $d_2 > d_1 > 0$ and $\Psi'(A_2) > \Psi'(A_1) > 0$, from (2.22) it follows that

$$|g'(A)| \le 2|A|\,\Psi'(A_2) + d_2, \quad A \in]-\infty, A_1[\ \cup\ ]A_2, +\infty[. \tag{2.23}$$

Since $g(A)$ is continuously differentiable, from (2.15) and (2.23)one gets (2.14). $\square$

3. Semidiscretization and scheme construction

The computation of numerical solutions of the model is necessary because an exact solution is not available. Among the more extended numerical techniques we should mention the finite-difference (FD) approach, [22,18]. The numerical analysis of the computed solution using FD methods for nonlinear models uses to be difficult and difficulties are overcome by means of linearization strategies that in some way falsify the model mainly near the maturity and the strike price. This fact motivates the search of an alternative numerical method which preserves the advantages of the FD method and that allows the full treatment of nonlinearities. The proposed method is the so called semidiscretization method (SD) or method of lines, [22–25]. By replacing the partial derivatives $\frac{\partial U}{\partial S}(S_i,\tau)$ and $\frac{\partial^2 U}{\partial S^2}(S_i,\tau)$ by finite-differences approximations [22,23,26], one gets the operators ∇_i and Δ_i defined by

$$\frac{\partial U}{\partial S}(S_i,\tau) = \nabla_i(\tau) + O(h^4), \tag{3.1}$$

$$\nabla_i(U(\tau)) = \frac{U(S_{i-2},\tau) - 8U(S_{i-1},\tau) + 8U(S_{i+1},\tau) - U(S_{i+2},\tau)}{12h}, \tag{3.2}$$

$$\frac{\partial^2 U}{\partial S^2}(S_i,\tau) = \Delta_i(\tau) + O(h^4), \tag{3.3}$$

$$\Delta_i(U(\tau)) = \frac{-U(S_{i-2},\tau) + 16U(S_{i-1},\tau) - 30U(S_i,\tau) + 16U(S_{i+1},\tau) - U(S_{i+2},\tau)}{12h^2}, \quad 0 \le \tau \le T, \tag{3.4}$$

where $S_j = E - L + jh$, $1 \le j \le N-1$ are the nodes of the underlying asset interval $[E-L, E+L]$, E is the strike price and $L \le E$ is the radius of the neighborhood about E where the numerical solution is computed. Let $u_i(\tau)$ be the approximation of theoretical value $U(S_i,\tau)$ and let us denote $u(\tau) = [u_1, u_2, \ldots, u_{N-1}]^T$, then the partial differential equation (1.6) is approximated by the system of ordinary differential equations

$$\frac{\mathrm{d}u(\tau)}{\mathrm{d}\tau} = B(\tau)u(\tau) + w(\tau), \tag{3.5}$$

where

$$B(\tau) = \begin{bmatrix} \gamma_1 & \delta_1 & \xi_1 & 0 & 0 & 0 & 0 & \ldots & 0 \\ \beta_2 & \gamma_2 & \delta_2 & \xi_2 & 0 & 0 & 0 & \ldots & 0 \\ \alpha_3 & \beta_3 & \gamma_3 & \delta_3 & \xi_3 & 0 & 0 & \ldots & 0 \\ 0 & \alpha_4 & \beta_4 & \gamma_4 & \delta_4 & \xi_4 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & \alpha_{N-3} & \beta_{N-3} & \gamma_{N-3} & \delta_{N-3} & \xi_{N-3} \\ 0 & \ldots & 0 & 0 & 0 & \alpha_{N-2} & \beta_{N-2} & \gamma_{N-2} & \delta_{N-2} \\ 0 & \ldots & 0 & 0 & 0 & 0 & \alpha_{N-1} & \beta_{N-1} & \gamma_{N-1} \end{bmatrix}, \tag{3.6}$$

$$w(\tau) = \begin{bmatrix} \alpha_1 u_{-1} + \beta_1 u_0 \\ \alpha_2 u_0 \\ 0 \\ \vdots \\ 0 \\ \xi_{N-2} u_N \\ \delta_{N-1} u_N + \xi_{N-1} u_{N+1} \end{bmatrix}, \tag{3.7}$$

and the entries functions of τ are defined by

$$\left.\begin{aligned}
\alpha_i &= \alpha_i(\tau) = -\frac{\sigma_i(\tau)^2 S_i^2}{24h^2} + \frac{rS_i}{12h}\\
\beta_i &= \beta_i(\tau) = \frac{2\sigma_i(\tau)^2 S_i^2}{3h^2} - \frac{2rS_i}{3h}\\
\gamma_i &= \gamma_i(\tau) = -\frac{15\sigma_i(\tau)^2 S_i^2}{12h^2} - r\\
\delta_i &= \delta_i(\tau) = \frac{2\sigma_i(\tau)^2 S_i^2}{3h^2} + \frac{2rS_i}{3h}\\
\xi_i &= \xi_i(\tau) = -\frac{\sigma_i(\tau)^2 S_i^2}{24h^2} - \frac{rS_i}{12h}
\end{aligned}\right\}, \tag{3.8}$$

being

$$\sigma_i^2 = \sigma_0^2(1+\Psi_i), \tag{3.9}$$

the square of volatility approximation at the node S_i and time τ, and being

$$\Psi_i = \Psi(a^2 S_i^2 \Delta_i(u(\tau))\mathrm{e}^{r\tau}), \tag{3.10}$$

the corresponding approximation of the volatility correction function. Boundary values u_{-1}, u_0, u_N and u_{N+1} appearing in (3.7) are computed by using fourth-order Lagrange interpolating polynomial passing throughout the four closest internal mesh points:

$$\begin{aligned}
&u_{-1} = 10u_1 - 20u_2 + 15u_3 - 4u_4,\\
&u_0 = 4u_1 - 6u_2 + 4u_3 - u_4,\\
&u_N = 4u_{N-1} - 6u_{N-2} + 4u_{N-3} - u_{N-4},\\
&u_{N+1} = 10u_{N-1} - 20u_{N-2} + 15u_{N-3} - 4u_{N-4}.
\end{aligned} \tag{3.11}$$

Taking into account (3.5)–(3.11), the semidiscretized approximating problem takes the form

$$\left.\begin{aligned}
&\frac{\mathrm{d}u}{\mathrm{d}\tau} = M(\tau)u(\tau), \quad 0 \le \tau \le T,\\
&u(0) = (u_1(0), \ldots, u_{N-1}(0))^t, \qquad u_i(0) = \max(S_i - E, 0), \quad 1 \le i \le N-1
\end{aligned}\right\}. \tag{3.12}$$

And $M(\tau)$ is given by

$$M(\tau) = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & 0 & \dots & \dots & 0\\
a_{21} & a_{22} & a_{23} & a_{24} & 0 & \ddots & \ddots & \vdots\\
\alpha_3 & \beta_3 & \gamma_3 & \delta_3 & \xi_3 & 0 & \ddots & \vdots\\
0 & \alpha_4 & \beta_4 & \gamma_4 & \delta_4 & \xi_4 & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0\\
\vdots & \dots & 0 & \alpha_{N-3} & \beta_{N-3} & \gamma_{N-3} & \delta_{N-3} & \xi_{N-3}\\
\vdots & \dots & \dots & 0 & a_{N-2N-4} & a_{N-2N-3} & a_{N-2N-2} & a_{N-2N-1}\\
0 & \dots & \dots & 0 & a_{N-1N-4} & a_{N-1N-3} & a_{N-1N-2} & a_{N-1N-1}
\end{bmatrix}, \tag{3.13}$$

where the nonzero entries of the first, second, last but one and last rows are given by

$$\begin{aligned}
&a_{11} = \gamma_1 + 10\alpha_1 + 4\beta_1 && a_{12} = \delta_1 - 20\alpha_1 - 6\beta_1\\
&a_{13} = \xi_1 + 15\alpha_1 + 4\beta_1 && a_{14} = -4\alpha_1 - \beta_1\\
&a_{21} = \beta_2 + 4\alpha_2 && a_{22} = \gamma_2 - 6\alpha_2\\
&a_{23} = \delta_2 + 4\alpha_2 && a_{24} = \xi_2 - \alpha_2\\
&a_{N-2N-4} = \alpha_{N-2} - \xi_{N-2} && a_{N-2N-3} = \beta_{N-2} + 4\xi_{N-2}\\
&a_{N-2N-2} = \gamma_{N-2} - 6\xi_{N-2} && a_{N-2N-1} = \delta_{N-2} + 4\xi_{N-2}\\
&a_{N-1N-4} = -\delta_{N-1} - 4\xi_{N-1} && a_{N-1N-3} = \alpha_{N-1} + 4\delta_{N-1} + 15\xi_{N-1}\\
&a_{N-1N-2} = \beta_{N-1} - 6\delta_{N-1} - 20\xi_{N-1} && a_{N-1N-1} = \gamma_{N-1} + 4\delta_{N-1} + 10\xi_{N-1}.
\end{aligned} \tag{3.14}$$

Using Euler method

$$u((n+1)k) = (I + M(nk))u(nk), \quad 0 \le n \le l, \tag{3.15}$$

the numerical solution of the vector initial value problem (3.12) one gets

$$u(\tau) = \left[\prod_{n=l-1}^{n=0} (I + kM(nk)) \right] u(0), \tag{3.16}$$

where $k = \Delta\tau,\ lk = \tau$, and the entries of matrix $M(\tau)$ are given by (3.8).

In the following we will use the notation

$$\Psi_i^n(u) = \Psi(e^{nkr} a^2 S_i^2 \Delta_i^n(u)), \tag{3.17}$$

and thus

$$\sigma_i^2(nk) = \sigma_0^2(1 + \Psi_i^n(u)). \tag{3.18}$$

From (3.15) and (3.18) one gets the FD scheme

$$F(u_j^n) = \frac{u_j^{n+1} - u_j^n}{k} - \frac{S_j^2}{2}\sigma_0^2(1 + \Psi_i^n(u))\Delta_j^n(u) - rS_j\nabla_j^n(u) + ru_j^n, \tag{3.19}$$

where $u_j^n = u(S_j, nk)$.

4. Stability

In this section we address the conditional time stability of the FD scheme (3.16) in the sense that for a fixed $h = \Delta S > 0$, the solution $u(\tau)$ remains bounded as $k = \Delta\tau$ tends to zero, $l \to \infty$ but with $lk = \tau$. This is the concept of stability in the fixed station sense with respect to time. Note that as h is fixed the size of the matrix $M(\tau)$ is also fixed. In other case, the size $N - 1$ would increase to infinity as h tends to zero.

Let $\rho = nk$ with $0 \le n \le l - 1$ and note that $M(\rho)$ given by (3.13) can be expressed in the form

$$M(\rho) = \frac{1}{24h^2}\Sigma(\rho)D^2P + \frac{r}{12h}DQ - rI, \tag{4.1}$$

where D and $\sum(\rho)$ are diagonal matrices, I is the identity matrix of size $N - 1$ with

$$\Sigma(\rho) = \sigma_0^2 \operatorname{diag}\left(1 + \Psi_1^n(u), 1 + \Psi_2^n(u), \dots, 1 + \Psi_{N-1}^n(u)\right), \tag{4.2}$$

$$D = \operatorname{diag}(S_1, S_2, \dots, S_{N-1}), \qquad E - L < S_i < E + L, \qquad 1 \le i \le N - 1, \tag{4.3}$$

and $P,\ \ Q$ are the constant matrices defined by

$$P = \begin{bmatrix} 24 & -60 & 48 & -12 & 0 & \cdots & 0 \\ 12 & -24 & 12 & 0 & 0 & \cdots & 0 \\ -1 & 16 & -30 & 16 & -1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & -1 & 16 & -30 & 16 & -1 \\ 0 & \cdots & 0 & 0 & 12 & -24 & 12 \\ 0 & \cdots & 0 & -12 & 48 & -60 & 24 \end{bmatrix}, \tag{4.4}$$

$$Q = \begin{bmatrix} -22 & 36 & -18 & 4 & 0 & \cdots & 0 \\ -4 & -6 & 12 & -2 & 0 & \cdots & 0 \\ 1 & -8 & 0 & 8 & -1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 1 & -8 & 0 & 8 & -1 \\ 0 & \cdots & 0 & 2 & -12 & 6 & 4 \\ 0 & \cdots & 0 & -4 & 18 & -36 & 22 \end{bmatrix}. \tag{4.5}$$

Let $P_0 = P_0(24, -24, -30, -30, \dots, -30, -24, 24)$ be the diagonal matrix having the same principal diagonal as P, let $P_1 = P_1(-60, 12, 16, \dots, 16, 12)$ be the matrix having zeroes everywhere with the exception of the first upper diagonal, that coincides with the first upper diagonal of P. Let $P_2 = P_2(48, 0, -1, -1, \dots, -1)$ be the matrix having zeroes everywhere with the exception of the second upper diagonal that coincides with the corresponding of P. Finally, let $P_3 = P_3(-12, 0, \dots, 0)$ the matrix having the third upper diagonal the same as the one of P, and zeroes outside of this upper diagonal. In an analogous way, from (4.4) one defines the matrices P_{-3}, P_{-2} and P_{-1} for the corresponding lower diagonals. Thus we may express P as the sum of seven sparse matrices in $\mathbb{R}^{N-1\times N-1}$ having only one nonzero diagonal

$$P = P_{-3} + P_{-2} + P_{-1} + P_0 + P_1 + P_2 + P_3. \tag{4.6}$$

From (4.5) and (4.6) we have

$$\|P\| \le \sum_{S=-3}^{3} \|P_S\| = 12 + 48 + 60 + 30 + 60 + 48 + 12 = 270. \tag{4.7}$$

By decomposing matrix Q in seven matrices in an analogous way to the one of matrix P, one gets

$$\|Q\| \le 4 + 18 + 36 + 22 + 36 + 18 + 4 = 138. \tag{4.8}$$

Note that from (3.4) one gets

$$\begin{aligned} |\Delta_i^n(u)| &\le \left(\frac{1+16+30+16+1}{12h^2}\right) \max\left\{|u_j^n|;\; 0 \le j \le N\right\}, \\ &= \frac{16}{3h^2} \max\left\{|u_j^n|;\; 0 \le j \le N\right\}, \quad 3 \le i \le N-3. \end{aligned} \tag{4.9}$$

For the remaining mesh points, from (3.4) and (3.11) it is easy to obtain that

$$|\Delta_i^n(u)| \le \left(\frac{12}{h^2}\right) \max\left\{|u_j^n|;\; 0 \le j \le N\right\}, \quad i = 1,\; i = N-1, \tag{4.10}$$

$$|\Delta_i^n(u)| \le \left(\frac{4}{h^2}\right) \max\left\{|u_j^n|;\; 0 \le j \le N\right\}, \quad i = 2,\; i = N-2. \tag{4.11}$$

From (4.9)–(4.11) it follows that

$$|\Delta_i^n(u)| \le \left(\frac{12}{h^2}\right) \max\left\{|u_j^n|;\; 0 \le j \le N\right\}. \tag{4.12}$$

From (3.17), (4.12) and Lemma 1, for $1 \le i \le N-1$ it follows that

$$\begin{aligned} &|\Psi_i^n(u)| \le L(n,h) \\ &L(n,h) = \Psi'(A_2)\mathrm{e}^{nkr} a^2 (E+L)^2 \frac{12}{h^2} \max\left\{|u_j^n|;\; 0 \le j \le N\right\} + d_2. \end{aligned} \tag{4.13}$$

From (4.2) and (4.13) and small enough values of h there exists a positive number $C_1(\rho)$ such that

$$\|\Sigma(\rho)\| \le \sigma_0^2 \left[\left(\frac{C_1(\rho)}{h^2}\right)\mathrm{e}^{nkr} + 1\right]. \tag{4.14}$$

From (4.1), (4.3), (4.7), (4.8), and (4.14) and taking into account $nk \le \tau$ it follows that

$$\|M(\rho)\| \le \frac{45}{4h^2}\sigma_0^2\left(1 + \frac{C_1(\rho)\mathrm{e}^{\tau r}}{h^2}\right)(E+L)^2 + \frac{23r}{2h}(E+L) + r \le \frac{C}{h^4}, \tag{4.15}$$

for small enough values of h.

From (4.15) and (3.16) and using that $kl = \tau$ one gets

$$\|u(\tau)\| \le \left(1 + k\frac{C(\tau)}{h^4}\right)^l \|u(0)\| \le \mathrm{e}^{\frac{\tau C(\tau)}{h^4}} \|u(0)\|. \tag{4.16}$$

Summarizing, from (4.16) one gets that the FD scheme (3.16) is conditionally time stable for appropriate fixed values of $h = \Delta S$.

5. Consistency

Dealing with reliable numerical computations of FD schemes, the consistency of the difference-scheme with the equation is a necessary requirement because this means that the exact theoretical solution of the partial differential equation approximates well to the exact solution of the difference equation as the stepsizes tend to zero, [23]. The strategy developed by the authors in [1, p. 383], of using a very small time step near the maturity is an advisable decision but by no means a guarantee that numerical results are reliable.

Let us represent Eq. (1.6) by $L(U) = 0$, and let $F(u_j^n) = 0$ represent the approximating difference equation defined by (3.19) with exact solution u. In accordance with [23, p.100], the FD scheme is consistent with (1.6) if

$$T_j^n(U) = F(U_j^n) - L(U_j^n) \;\to\; 0, \quad \text{as } h = \Delta S \;\to\; 0,\; k = \Delta t \to\; 0, \tag{5.1}$$

where U_j^n denotes the theoretical solution of (1.6) evaluated at the point $(S_j,\ nk)$, i.e., $U_j^n = U(S_j, nk)$. If in (5.1) one has that $T_j^n(U) = O(h^p) + O(k^q)$, then we say that the FD scheme is consistent of order (p, q).

Taking into account (3.19), (3.2) and (3.4) for the internal points we have

$$F(U_j^n) = \frac{U_j^{n+1} - U_j^n}{k} - \frac{S_j^2}{2}\sigma_0^2(1 + \Psi_j^n(U))\Delta_j^n(U) - rS_j\nabla_j^n(U) + rU_j^n, \tag{5.2}$$

where

$$\Psi_j^n(U) = \Psi\left(\mathrm{e}^{nkr}a^2\Delta_j^n(U)\right).$$

Assuming that U admits partial derivatives with respect to S up order six, and using Taylor expansion about (S_j, nk) one gets

$$\begin{aligned}\nabla_j^n(U) &= \frac{\partial U}{\partial S}(S_j, nk) + \frac{h^4}{180}\left\{\frac{\partial^5 U}{\partial S^5}(\xi_1, nk) - \frac{\partial^5 U}{\partial S^5}(\xi_{-1}, nk) + 4\frac{\partial^5 U}{\partial S^5}(\xi_{-2}, nk) - 4\frac{\partial^5 U}{\partial S^5}(\xi_2, nk)\right\}\\ &= \frac{\partial U}{\partial S}(S_j, nk) + h^4E_j^n(1),\end{aligned} \tag{5.3}$$

where

$$\xi_1 \in]S_j, S_j + h[;\ \xi_{-1} \in]S_j - h, S_j[;\ \xi_{-2} \in]S_j - 2h, S_j[;\ \xi_2 \in]S_j, S_j + 2h[;$$

and

$$\mid E_j^n(1) \mid \leq \frac{1}{18} \mid U^n(1) \mid_{\max}, \tag{5.4}$$

$$\mid U^n(1) \mid_{\max} = \max\left\{\left|\frac{\partial^5 U}{\partial S^5}(S, nk)\right|\ ;\ E - L \leq S \leq E + L\right\}, \tag{5.5}$$

$$\begin{aligned}\Delta_j^n(U) &= \frac{\partial^2 U}{\partial S^2}(S_j, nk) + \frac{h^4}{90}\left\{\frac{\partial^6 U}{\partial S^6}(\eta_1, nk) - 4\frac{\partial^6 U}{\partial S^6}(\eta_2, nk)\right\}\\ &= \frac{\partial^2 U}{\partial S^2}(S_j, nk) + h^4E_j^n(2),\end{aligned} \tag{5.6}$$

where

$$\eta_1 \in]S_j - h, S_j + h[\ ;\ \eta_2 \in]S_j - 2h, S_j + 2h[,$$

$$|E_j^n(2)| \leq \frac{1}{18}|U^n(2)|_{\max}, \tag{5.7}$$

$$|U^n(2)|_{\max} = \max\left\{\left|\frac{\partial^6 U}{\partial S^6}(S, nk)\right|\ ;\ E - L \leq S \leq E + L\right\}, \tag{5.8}$$

$$\frac{U_j^{n+1} - U_j^n}{k} = \frac{\partial U}{\partial \tau}(S_j, nk) + kE_j^n(3), \tag{5.9}$$

with

$$E_j^n(3) = \frac{1}{2}\frac{\partial^2 U}{\partial \tau^2}(S_j, \delta), \quad \delta \in]nk, (n+1)k[\tag{5.10}$$

$$|E_j^n(3)| \leq \frac{1}{2}|U_j(1)|_{\max} = \frac{1}{2}\max\left\{\left|\frac{\partial^2 U}{\partial \tau^2}(S_j, \tau)\right|\ ;\ nk \leq \tau \leq (n+1)k\right\}. \tag{5.11}$$

From (5.1), (5.2), (5.3), (5.6) and (5.9) it follows that

$$\begin{aligned}T_j^n(U) &= F(U_j^n) - L(U_j^n)\\ &= -\frac{S_j^2}{2}\sigma_0^2\left\{\left(1 + \Psi_j^n(U)\right)\Delta_j^n(U) - \left(1 + \Psi\left(\mathrm{e}^{nkr}a^2\frac{\partial^2 U}{\partial S^2}(S_j, nk)\right)\right)\frac{\partial^2 U}{\partial S^2}(S_j, nk)\right\}\\ &\quad + kE_j^n(3) - rS_jE_j^n(1)h^4.\end{aligned} \tag{5.12}$$

Let us denote

$$A_j^n = \mathrm{e}^{nkr}a^2S_j^2\frac{\partial^2 U}{\partial S^2}(S_j, nk), \tag{5.13}$$

$$\Delta A_j^n = \mathrm{e}^{nkr}a^2S_j^2E_j^n(2)h^4, \tag{5.14}$$

and note that using the function g introduced in Lemma 1–(iv), and (5.6), one gets

$$\left\{\left(1+\Psi_j^n(U)\right)\Delta_j^n(U)-\left(1+\Psi\left(\mathrm{e}^{nkr}a^2\frac{\partial^2 U}{\partial S^2}(S_j,nk)\right)\right)\frac{\partial^2 U}{\partial S^2}(S_j,nk)\right\}$$
$$=\left\{\left(1+\Psi(A_j^n+\Delta A_j^n)\right)(A_j^n+\Delta A_j^n)-\left(1+\Psi(A_j^n)\right)A_j^n\right\}\frac{1}{\mathrm{e}^{nkr}a^2S_j^2}$$
$$=\left\{g(A_j^n+\Delta A_j^n)-g(A_j^n)+\Delta A_j^n\right\}\mathrm{e}^{-nkr}a^{-2}S_j^{-2}. \tag{5.15}$$

By using the mean value theorem and Lemma 1–(iv) it follows that

$$g(A_j^n+\Delta A_j^n)-g\left(A_j^n\right)=g'(A_j^n+\theta(\Delta A_j^n))\,(\Delta A_j^n),\quad 0<\theta<1. \tag{5.16}$$

From (5.12)–(5.16), one gets that the truncation error $T_j^n(U)$ satisfies

$$T_j^n(U)=-\frac{S_j^2}{2}\sigma_0^2\left(1+g'A_j^n+\theta(\Delta A_j^n)\right)E_j^n(2)h^4-rS_jE_j^n(1)h^4+kE_j^n(3). \tag{5.17}$$

From (5.4), (5.7), (5.11), (5.17) and (2.14), (2.15) it follows that

$$|T_j^n(U)|\le\frac{h^4}{18}\left\{\frac{(E+L)^2}{2}\sigma_0^2(1+C_2(h))|U^n(2)|_{\max}+r(E+L)|U^n(1)|_{\max}\right\}+\frac{k}{2}|U_j(1)|_{\max}, \tag{5.18}$$

where

$$|U^n(3)|_{\max}=\max\left\{\left|\frac{\partial^2 U}{\partial S^2}(S,nk)\right|\ ;\ E-L\le S\le E+L\right\}, \tag{5.19}$$

and

$$C_2(h)=\max\left\{G,\mathrm{e}^{nkr}a^2(E+L)^2\left(|U^n(3)|_{\max}+\frac{h^4}{18}|U^n(2)|_{\max}\right)\Psi'(A_2)+d_2\right\}, \tag{5.20}$$

where G, A_2 and d_2 are introduced in Lemma 1. Summarizing, we have that the FD scheme (3.19) is consistent of order (4, 1), i.e,

$$T_j^n(U)=O(h^4)+O(k). \tag{5.21}$$

Thus the following result has been established:

Theorem 1. *Let $S=E-L+jh$, $0\le\tau=nk\le T$ with $h=\Delta S$, $k=\Delta\tau$ with $3\le j\le N-3$ and $2L=Nh$. Then the truncation error of FD scheme (3.19) and the point (S,τ) is consistent of order $(4,1)$ with Eq. (1.6)*

6. Examples

The first example, related to the case $a=0$, i.e., market without transaction costs, compares the computed numerical solution with the exact solution.

Example 1. Consider the vanilla call option with parameters

$$\sigma=0.2,\qquad r=0.04,\qquad E=40,\qquad \tau=0.5\text{ years}.$$

Fig. 1 shows how the difference between the exact solution and the numerical solution at $\tau=0.5$ years decrease as h decrease.

The next example shows the change of the price option with the parameter a.

Example 2. Consider the vanilla call option with transaction costs and parameters

$$\sigma=0.2,\qquad r=0.1,\qquad E=100,\qquad \tau=1\text{ year},\qquad h=4,\qquad k=0.005.$$

Fig. 2 shows option pricing valuation of this call option for several values of the parameter a as well as the pay-off function. The next example illustrates the time stability property of the numerical solution.

Example 3. With the parameters of example 2, taking a fixed value of h, one represents the difference

$$u(S,\tau,\Delta\tau=k)-u(S,\tau,\Delta\tau=k/2)$$

with parameters (Fig. 3)

$$\sigma=0.2,\qquad r=0.1,\qquad E=100,\qquad \tau=1\text{ year},\qquad h=10.$$

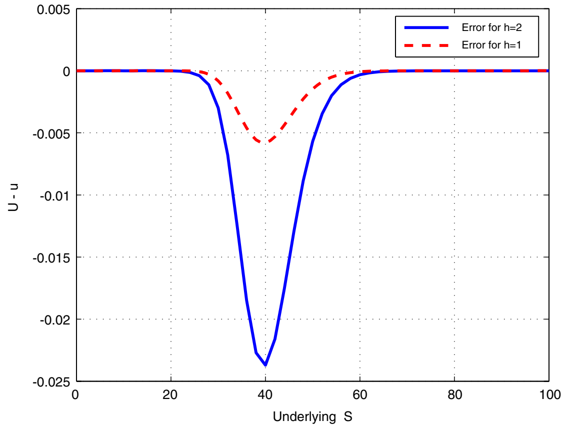

Fig. 1. Error of numerical solution for the linear case.

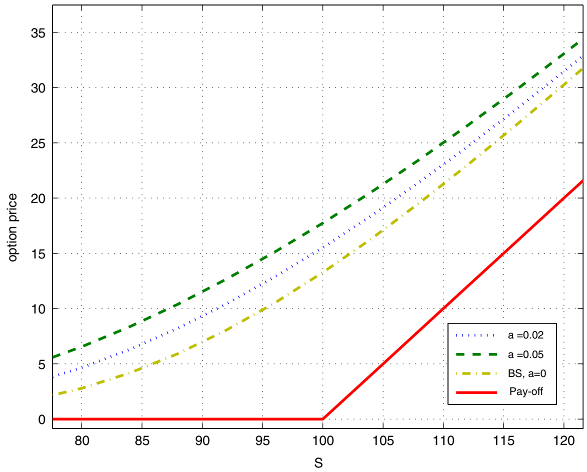

Fig. 2. Valuation of a vanilla call option in both linear and nonlinear cases.

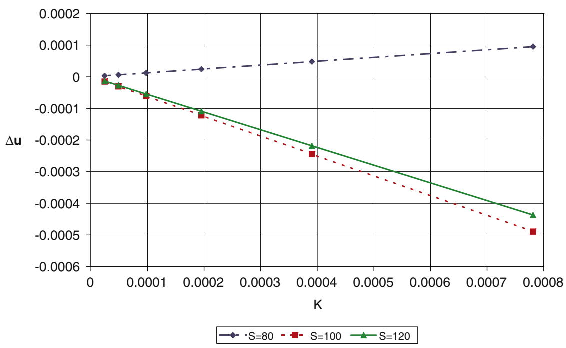

Fig. 3. Difference between u_k and $u_{k/2}$.

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